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DECOMPOSITIONS OF HOMOLOGY MANIFOLDS AND THEIR GRAPHS

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ABSTRACT
The graph of every d-dimensional convex polytope is d-connected and contains a refinement of the complete graph on $d+1$ vertices. These two theorems are generalized to pseudomanifolds and to some very general decompositions of homology manifolds.

1. Introduction

Two of the most fundamental properties of the graphs of convex d-dimensional polytopes are that they are d-connected and that they contain a refinement of the complete graph on $d+1$ vertices. These two properties have been shown to be true for the 1-skeletons of $(d+1)$-manifolds, as well [1]. In this paper we give much simpler proofs of these properties for manifolds. We also consider what happens if our manifold is decomposed into faces that are more complex than simplices. We show that for some purposes, homology manifolds are the best setting to work in, and by the use of structures called gchc's, we develope a type of decomposition of a homology manifold that always admits a dual decomposition. Included in these structures will be the triangulations of homology manifolds, and these structures will have the above two graph properties.

2. Preliminary definitions and lemmas

We shall be dealing with various kinds of simplicial complexes and also more general decompositions of cells, manifolds, and homology manifolds. The remarks in this section apply to all of these, so we shall simply refer to these as complexes.

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Two faces of a complex are said to be *incident* provided one contains the other. Two complexes are *isomorphic* provided there is a one-to-one, incidence preserving, dimension preserving, function taking the set of faces of one complex onto the set of faces of the other. The complexes are *dual* if there is a one-to-one incidence reversing function taking the set of faces of one complex onto the set of faces of the other.

There are two incidence properties that all of our complexes will have, and we shall refer to them simply as *the two incidence properties*. For any complex C they are:

(1) If F_1 is a face of C and if F_2 is a face of F_1, then F_2 is a face of C.

(2) If F_1 and F_2 are faces of C, then $F_1 \cap F_2$ is a face of both F_1 and F_2 (possibly the empty face).

If F is a face of a complex then the *star* of F, denoted star(F), is the set of all faces that contain F together with all faces of these faces. The *antistar* of F, denoted ast(F), is the set of all faces that miss F. The link of F, denoted link(F), is the intersection of the star and the antistar.

If our complex is d-dimensional, the faces of dimension d are called *facets* while the faces of dimension $d-1$ are called *subfacets*. By a *chain of facets* we mean a sequence of facets $F_1, \cdots, F_n$ such that the intersection of any two consecutive facets is a subfacet. The complex is *strongly connected* provided for each two facets there is a chain of facets beginning with one and ending with the other.

One complex S_1 is a *refinement* of a complex S_2 provided there is a homeomorphism of S_2 to S_1 such that the image of each face of S_2 is the union of faces of S_1. If the complex is a graph, then this definition is equivalent to the usual definition that S_1 is obtained from S_2 by adding vertices on edges of S_2. The vertices of S_1 that are images of vertices of S_2 are called the *principal vertices* of S_1.

The *graph* of a complex is the graph formed by the vertices and edges of the complex.

A pseudomanifold M is a finite simplicial d-complex such that

(i) M is strongly connected.

(ii) Every simplex in M belongs to a simplex of dimension d.

(iii) Every simplex of dimension $d-1$ belongs to exactly two simplices of dimension d.

One type of pseudomanifold that we shall investigate is the *homology manifold*. A homology d-manifold is a connected compact topological m-dimensional polyhedron on which the local homology groups at each point are isomorphic to the respective homology groups of the $(m-1)$-sphere. A *d-manifold* is a connected compact space such that every point has a neighborhood homeomorphic to an open d-dimensional cell. A *triangulated manifold* is a manifold that is the union of the simplices of a finite simplicial complex. The class of homology manifolds includes all manifolds.

3. Graphs of pseudomanifolds

The theorem that the graph of a d-polytope is d-connected is easily generalized to pseudomanifolds, but first we need two lemmas.

LEMMA 1. *If v is a vertex of a d-pseudomanifold then the strongly connected components of the link of v are $(d-1)$-pseudomanifolds.*

The proof is very simple and is left to the reader.

LEMMA 2. *Let S be a strongly connected simplicial d-complex such that every simplex belongs to a d-simplex. Then, the graph of S is d-connected.*

PROOF. Let u and v be two vertices of S and let $F_1, \cdots, F_n$ be a chain of facets joining them. Consider any set of vertices that separates the graph of this chain. If the intersection of each two consecutive facets in the sequence contains a vertex not in the separating set, then these vertices are the vertices of a path from u to v. It follows that the intersection of some two consecutive facets in the chain has all of its vertices in the separating set. The intersections are subfacets and thus have at least d vertices.

This lemma is a special case of a theorem by Sallee [7]. We included its proof here for completeness.

THEOREM 1. *The graph of a d-pseudomanifold M is $d+1$ connected.*

PROOF. This is clearly true if $d=1$ since then the pseudomanifold is a polygon. Proceeding by induction, let $v_1, v_2, \cdots, v_n$ be a minimal separating set of vertices for the graph of M. We consider the link of v_1. If the remaining vertices, $v_2, \cdots, v_n$ separate one of the strongly connected components of $\mathrm{link}(v_1)$ then by induction there are at least d of them separating this component and thus there are at least $d+1$ vertices in all. Suppose, now, that none of the strongly connected components of $\mathrm{link}(v_1)$ are separated. We construct a new pseudomanifold M' by removing v_1 and every face containing it and inserting new vertices, one for each strongly connected component of $\mathrm{link}(v_1)$, and also add to the complex simplices that are the join of a simplex of a strongly connected component of link(v_1) and its new vertex (see Fig. 1). The vertices $v_2, \cdots, v_n$ separate the graph of this new complex, for, observe that when we remove v from the graph of M, the remaining vertices separate the graph that remains. Each strongly connected component of link(v_1) determines a graph that lies in just one component of the separated graph. If there is only one strongly connected component of link(v_1), then v_1 is joined only to one component of the separated graph and thus $v_2, \cdots, v_n$ separates the pseudomanifold, contradicting the minimality of $v_1, \cdots, v_n$. If there are at least two connected components of link(v_1) then $v_2, \cdots, v_n$ separate the graph of M'.

It is easily checked that M' is strongly connected, and by Sallee's theorem it requires at least d vertices to separate it, thus $v_1, \cdots, v_n$ contains at least $d+1$ vertices.

This theorem immediately gives us the same result for homology manifolds and manifolds because they are all pseudomanifolds (see [5], ch. 7), however it is

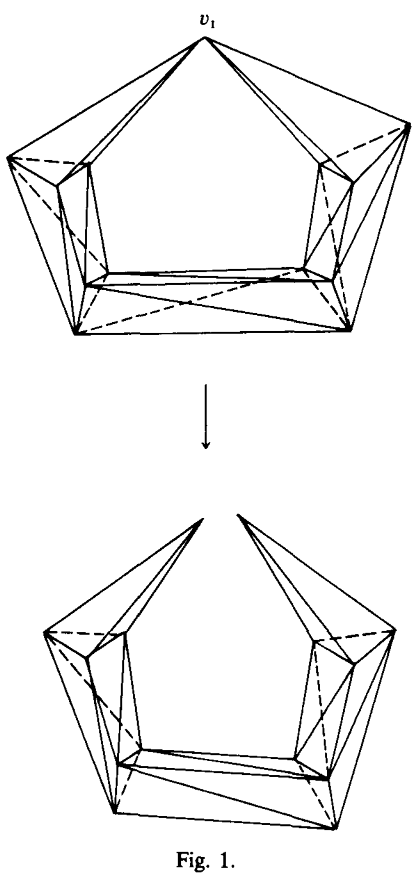

Fig. 1.

interesting to see how simple the proof becomes when one deals just with homology manifolds.

The nice property of homology manifolds that we use is that in any triangulation of the homology manifold, the link of each vertex is a homology sphere, and thus is a homology manifold of one lower dimension (see [4, ch. 2]). To prove the connectivity theorem in this setting, proceed as above with a minimal separating set of vertices. If $v_2, \cdots, v_n$ do not separate link(v_1) then v_1 is connected to only one component of the separated graph, thus $v_2, \cdots, v_n$ separate the graph, contradicting the minimality of the separating set. But if link(v_1) is separated then by induction it takes at least d vertices to do it and we are done. It is interesting to compare this short proof with the long cumbersome proof in [1].

The second graph theorem for polytopes is even easier to generalize to pseudomanifolds.

THEOREM 2. *Each vertex of a d-pseudomanifold is a principal vertex of a refinement of the complete graph on $d+2$ vertices.*

PROOF. The statement clearly is true for $d = 1$. Proceeding by induction, let v be a vertex of a d-pseudomanifold. The link of v has a strongly connected component that is a $(d-1)$-pseudomanifold. By induction this link contains a refinement of the complete graph on $d+1$ vertices. Since v is joined to each vertex of the link we are done.

There are several other graph theorems for polytopes (see [6] and [7]). One, which is easily stated, was proved by Klee [5].

THEOREM 3. *If n vertices separate the graph of a d-polytope then the number of components of the separated graph is at most*

(1) 2, *if $n = d$,*

(2) *the maximum number of facets of any d-polytope with n vertices if $n \geqq d+1$;*

furthermore, these bounds are sharp.

This third graph theorem for polytopes is very troublesome for pseudomanifolds. The author conjectures the following:

If the graph of a d-pseudomanifold is separated by n vertices then the number of components is at most

(i) *the maximum number of facets of any d-pseudomanifold with n vertices, if $n > d+1$ and n is less than the minimum number of vertices necessary to triangulate the pseudomanifold;*

(ii) *the maximum number of facets of any triangulation of the pseudomanifold with n vertices, when n is at least the minimum number of vertices necessary to triangulate the pseudomanifold.*

There is one case not covered in the above conjecture, and that is the one case we can establish.

THEOREM 4. *If the graph of a d-pseudomanifold is separated by $d+1$ vertices, then it is separated into exactly two components.*

PROOF. The theorem clearly holds for $d = 1$. Proceeding by induction, let v be a vertex of a separating set of $d+1$ vertices of a d-pseudomanifold. If some component of the separated graph did not meet link(v) then removing v from the separating set would produce a separating set of d vertices, contradicting the d-connectedness of the graph. By induction link(v) is separated into exactly two components thus there are exactly two components of the separated graph.

4. Duality in homology manifolds

All of the complexes that we have looked at so far have faces that are all simplices. If one wishes to generalize polytopes one would want the faces to have a wider range of structures. To do this we introduce *generalized combinatorial cells* (abbreviated gcc) and *generalized combinatorial homology cells* (abbreviated gchc).

A (-1)-gcc (gchc) is the empty set. A 0-gcc (gchc) is a point. Inductively, a d-gcc (gchc), C, is a closed d-cell (d-homology cell) whose boundary is the union of k-gcc's (gchc's), called *faces*, such that the two incidence properties hold.

It is easily seen that any 2-gcc or 2-gchc is isomorphic to a polygon. It follows from a theorem of Steinitz [8] that any 3-gcc or 3-gchc is isomorphic to a 3-polytope. In four dimensions, however, there are gcc's that are not isomorphic to 4-polytopes (see [2]). Many properties of gcc's are found in [3].

While the topologist loves to build things out of simplices, and once in a while cubes, the geometer may use a wider variety of building blocks through the use of gcc or gchc complexes. A d-gcc-*complex* is a collection of k-gcc's, $-1 \leqq k \leqq d$, called faces of the complex, such that the two incidence properties hold. Substitute "gchc" for "gcc" and you have the definition of a d-gchc-*complex*. If the union of the faces of a d-gchc-complex is a manifold, homology manifold or pseudomanifold we call it a d-gchc *decomposition* of that manifold, homology manifold or pseudomanifold. The d-gchc decompositions of homology manifolds are particularly nice because they all have dual d-gchc decompositions, a fact that we shall now establish.

We shall need a few facts about homology spheres.

LEMMA 3. *The join of a point and a homology d-sphere or a homology d-cell is a homology $(d+1)$-cell.*

LEMMA 4. *The antistar of a vertex in a triangulated homology sphere is a homology cell.*

The first lemma is a standard theorem of homology theory. The second is an easy application of Mayer–Vietoris sequences using the fact that the star of a vertex is a homology cell and the link is a homology sphere.

LEMMA 5. *If v is a vertex of a d-gchc C then there is a triangulation of C having the same vertices as C and such that the star of v in C and in the triangulation of C are the same set.*

PROOF. Our proof is by induction on the dimension of C. For $d = 1$ the statement is clearly true. If C is a d-gchc, $d > 1$, then by induction we may triangulate the antistar of v in C using only vertices of C. Now we triangulate C by taking the join of v with each face in the antistar of v. By Lemma 3 this join is homeomorphic to star(v), which it replaces.

LEMMA 6. *If v is a vertex of a d-gchc-complex C, then there is a triangulation of C using only vertices of C such that the star of v in C and the star of v in the triangulation are the same set.*

PROOF. Our proof is by induction on the number of vertices of C. By induction we can triangulate the antistar of v. We obtain the desired triangulation by taking the join of v and faces in the induced triangulation of the link of v.

From the previous lemma we immediately get

THEOREM 5. *The link of a vertex in a d-gchc-decomposition of a homology manifold is a homology sphere.*

PROOF. The link of a vertex in any triangulation of the homology manifold is a homology sphere, but the link in the d-gchc decomposition is the same set as the link in a triangulation.

The dual of certain gchc-complexes will be constructed through the use of barycentric subdivisions. Each face F of a d-gchc-complex C will be associated with a point p_F, called its barycenter. We partially order the barycenters by $p_F < p_G$ if and only if F is a face of G. The simplices of the *barycentric subdivision* of C are the simplices determined by the linearly ordered sets of barycenters. It is an easy exercise to prove that the barycentric subdivision is homeomorphic to C.

The barycentric star *dual to a face* F of C is defined to be the set of all simplices of the barycentric subdivision, whose last vertex is the barycenter of F.

If v is a vertex of a d-gchc-complex C then by Lemma 4 the antistar of v in any face meeting v is a homology cell. The *vertex figure* of v in G is the gchc-complex consisting of all such antistars.

THEOREM 6. *The set of barycentric stars of a d-gchc-decomposition of a homology manifold M is a d-gchc-decomposition of M, that is dual to the original decomposition. The facets of the dual decomposition are dual to the vertex figures of the vertices of the original decomposition.*

PROOF. Our proof is by induction on the dimension of the homology manifold. For $d \leqq 2$ the theorem is well known. For $d > 2$, let v be any vertex of our manifold and let S be its barycentric star. For any barycenter p_F of a face F meeting v, let A_F be the antistar of v in F. We now consider the barycentric subdivision of the boundary of S. For each face A_F of S let q_F be its barycenter. We now have a one-to-one correspondence between the vertices of the boundary of S and the vertices of the barycentric subdivision of the vertex figure. We now show that this correspondence is an isomorphism. Suppose that $p_{F1}, p_{F2}, \cdots, p_{Fk}$ is a linearly ordered set of vertices determining a face in the boundary of S. Then Fi is a face of $F(i+1)$ for all $1 \leqq i \leqq k-1$. It then follows that A_{Fi} is a face of $A_{F(i+1)}$ in the vertex figure for all $1 \leqq i \leqq k-1$, thus $q_{F1}, \cdots, q_{Fk}$ determines a simplex in the barycentric subdivision of the vertex figure. This shows that a set of vertices determines a face in the barycentric subdivision of the vertex figure if the corresponding vertices determine a face in the boundary of S. The converse is proved in the same way and it follows that the two complexes are isomorphic.

As sets, the vertex figure and the link of v in M are the same, thus the vertex figure is a homology sphere. From the above, it follows that the barycentric star dual to the vertex is a homology cell.

Consider any face F meeting v. Its barycentric star will be a subcomplex of the boundary of S, and it will correspond to the barycentric star dual to the face A_F in the barycentric subdivision of the vertex figure, through the one-to-one correspondence we have set up for vertices. Since our choice of v was arbitrary, it follows that all barycentric stars in the barycentric subdivision of M are homology cells.

It is easily checked that the set of barycentric stars satisfies the two incidence properties and that incidences of the stars are the reverse of the incidences of the original faces of M, thus the set of barycentric stars forms a dual gchc decomposition of M. The above-mentioned correspondence between the stars dual to the faces meeting v in M, and the barycentric stars in the subdivision of the vertex figure shows that the facets of the dual gchc decomposition are dual to the vertex figures.

In order to get our graph theorems for gchc decompositions of homology manifolds we use the vertex figures instead of the links of vertices. It is immediate that the link of a vertex v is a refinement of the vertex figure of v and that the principal vertices of the refinement are just the vertices joined to v, thus we may use the fact that the link contains a refinement of the graph of a homology sphere and proceed by induction as we did in the previous theorems. We therefore have

THEOREM 7. *The graph of a* d-gchc *decomposition of a homology manifold is* $(d+1)$-*connected and contains a refinement of the complete graph on* $d+2$ *vertices.*

In particular, the graph of a d-gchc is d-connected, since its boundary is a $(d-1)$-gchc-sphere. From this one can generalize Sallee's theorem:

THEOREM 8. *If C is a strongly connected* d-gchc-*complex in which each face belongs to a d-face, then the graph of C is d-connected.*

REMARKS. (1) Manifolds with boundaries do not provide as rich a hunting ground for new graph theorems as one might think. By Sallee's theorem a triangulated d-manifold with boundary is d-connected and easy examples show that this is best possible. One can prove, using our techniques of looking at the graph of the link, that if d vertices separate the graph of a d-manifold with boundary, then all separating vertices lie on the boundary.

(2) Perhaps the ultimate complex would be a d-gchc-pseudomanifold, i.e., a strongly connected d-gchc-complex in which each $(d-1)$-face belongs to exactly two d-faces. A slight modification of our methods will prove that these, too, have the two graph properties of pseudomanifolds.

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